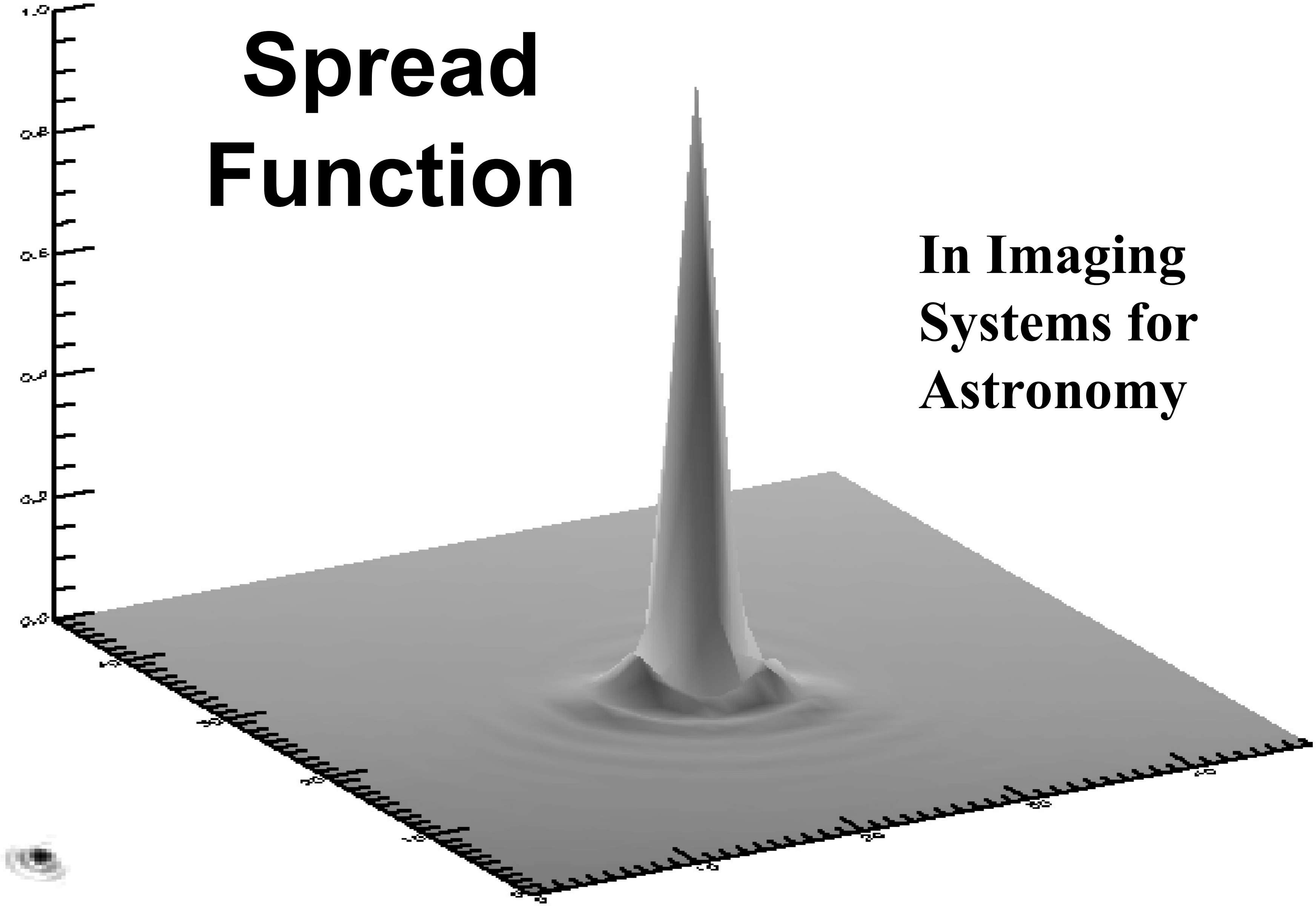
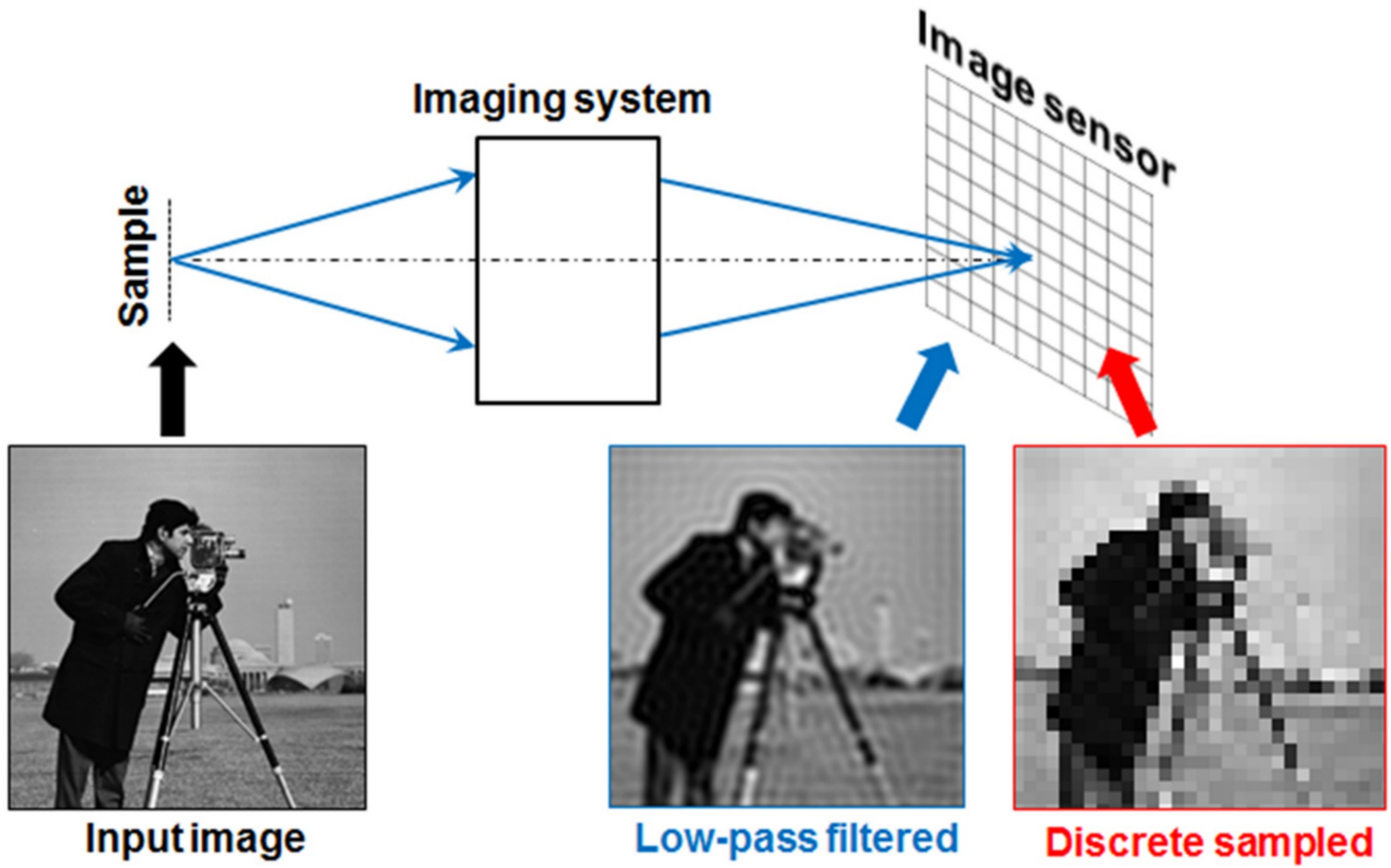
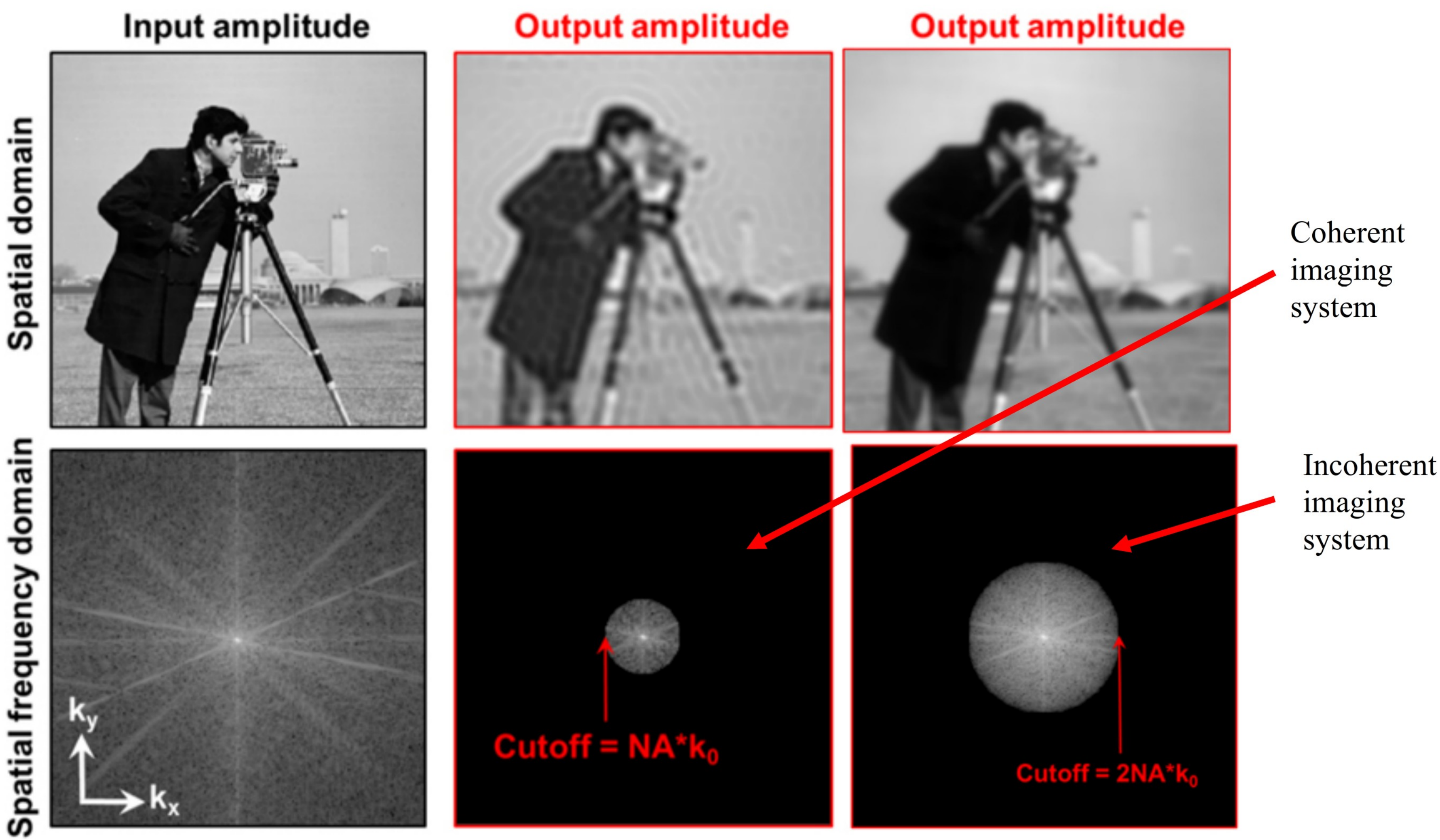
The Point Spread Function



In Imaging Systems for Astronomy


Imaging system
Image sensor
Sample
Input image
Low-pass filtered
Discrete sampled

Imaging Systems as Filters



Imaging Systems as Filters

Fraunhofer diffraction:
- plane waves
- FAR field
- ~Paraxial

Monochromatic (almost)

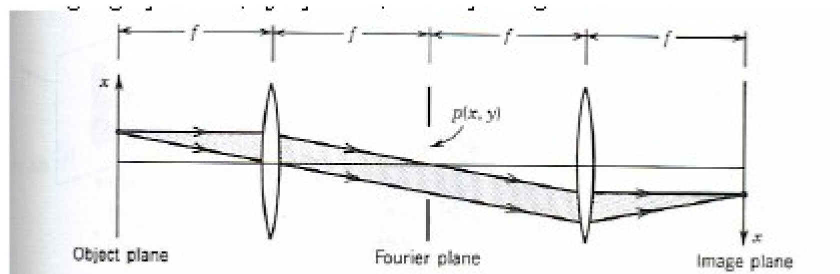


Figure 4.4-3 The 4-f imaging system. If an inverted coordinate system is used in the image plane, the magnification is unity.

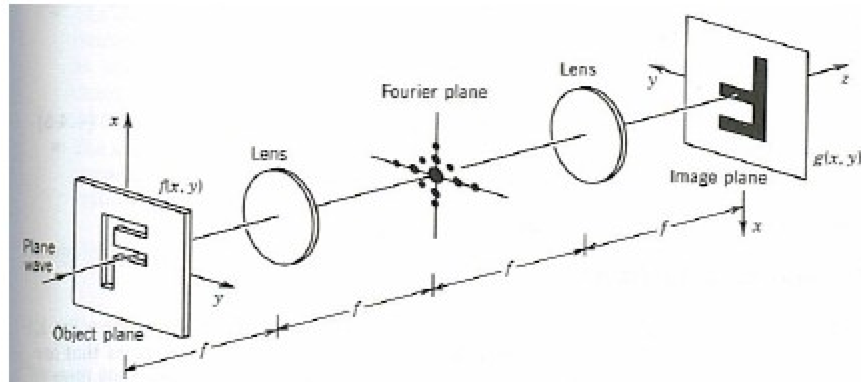


Figure 4.4-4 The 4-f system performs a Fourier transform followed by an inverse Fourier transform, so that the image is a perfect replica of the object.

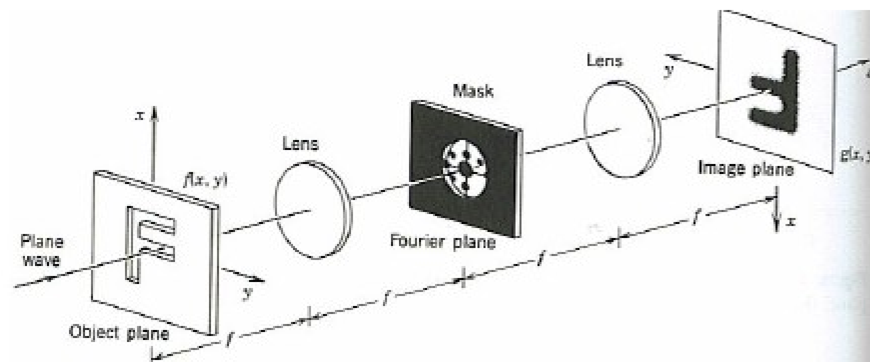
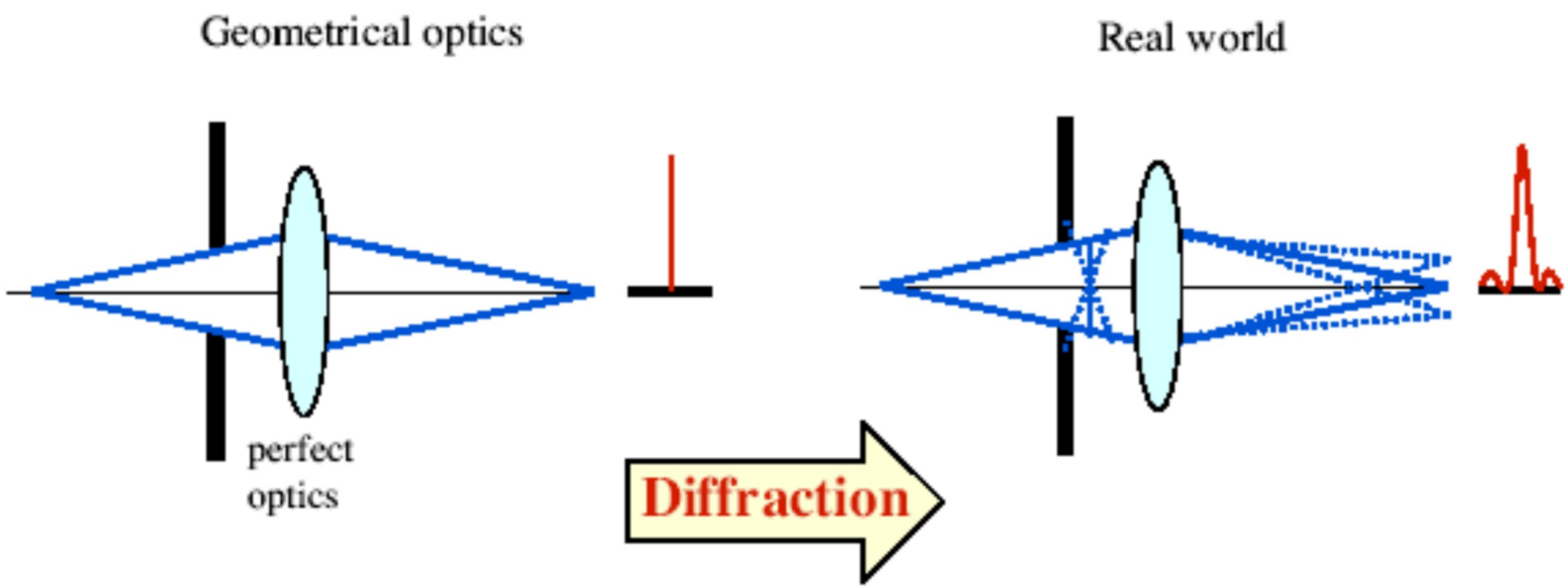
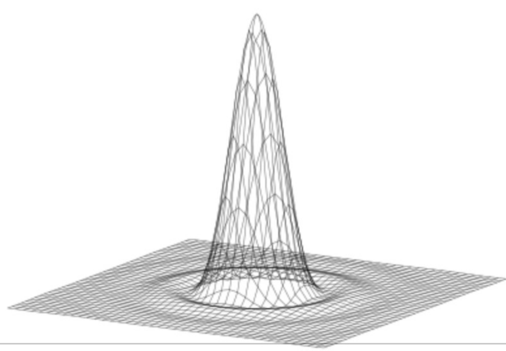


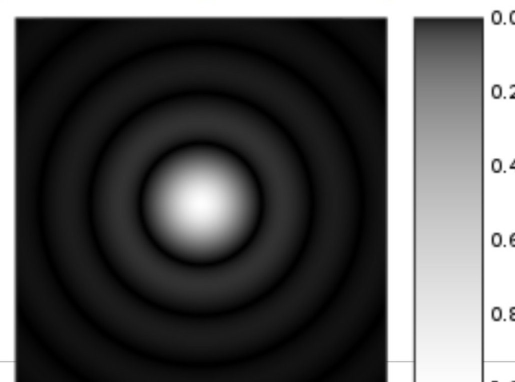
Figure 4.4-5 Spatial filtering. The transparencies in the object and Fourier planes have complex amplitude transmittances $f(x, y)$ and $p(x, y)$. A plane wave traveling in the z direction is modulated by the object transparency, Fourier transformed by the first lens, multiplied by the transmittance of the mask in the Fourier plane and inverse Fourier transformed by the second lens. As a result, the complex amplitude in the image plane $g(x, y)$ is a filtered version of $f(x, y)$. The system has a transfer function $\mathcal{H}(\nu_x, \nu_y) = p(\lambda f \nu_x, \lambda f \nu_y)$.

Point Spread Function (PSF)



The PSF for a perfect optical system is the Airy disc, which is the Fraunhofer diffraction pattern for a circular pupil.

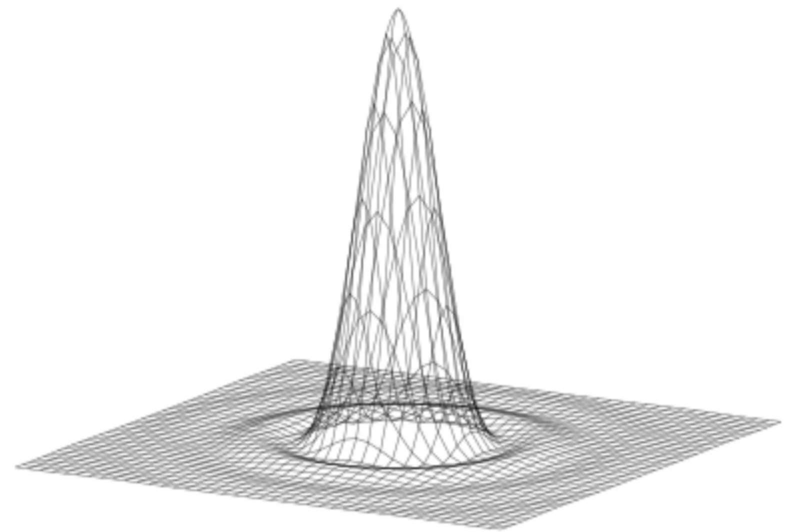




$s(x,y;x',y')$ = the image of a point source on the focal plane.

The image of a point source in an **ideal instrument** is defined by the diffraction pattern of the entrance pupil.
For a circular pupil, it is named the Airy function:

$$P_0(\vec{\alpha})=\left(\frac{\pi D^2}{4\lambda^2}\right)\left[\frac{2J_1(\pi D|\vec{\alpha}|)/\lambda}{D|\vec{\alpha}|/\lambda}\right]^2$$



where:
$P_0(\alpha)$ is the intensity on the focal plane,
as a function of the angular coordinate $\boldsymbol{\alpha}$;
λ is the light wavelength;
D is the entrance diameter of the telescope;
J_1 is a Bessel function.

The first zero is located at an angular distance $1.22\lambda/D$ from the maximum.
That distance is often used as a describer of the limit resolution for telescopes (*Rayleigh's criterion*).

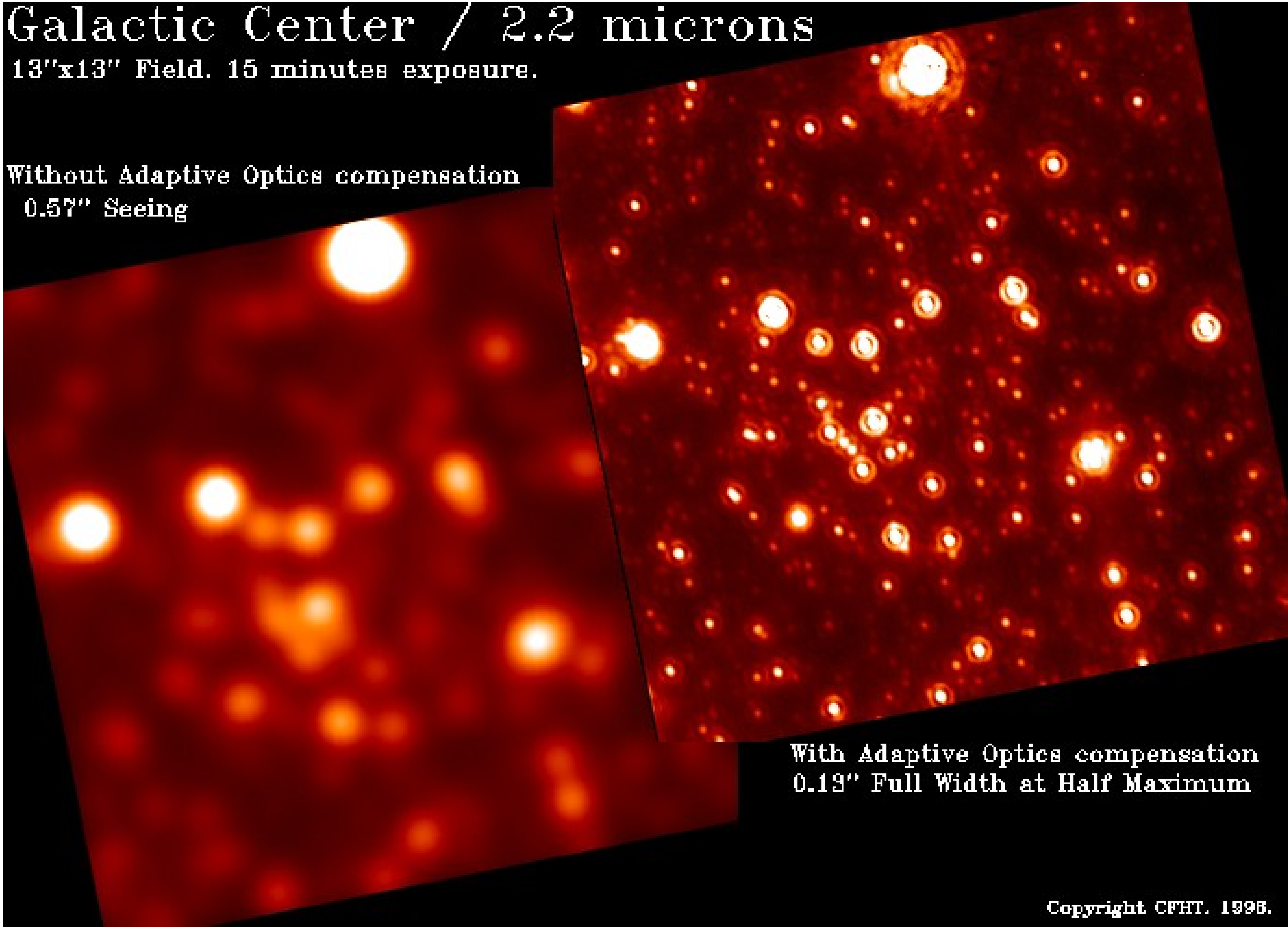
Every resolved object *O(α)* can be considered as a composed by many point sources. Any point source generates its PSF. The image of the object *I(α)* is given by the **convolution** of the object with the PSF.

$$I(\vec{\alpha}) = \int O(\vec{\beta}) P_0(\vec{\alpha} - \vec{\beta}) d\vec{\beta} = O \otimes P_0$$

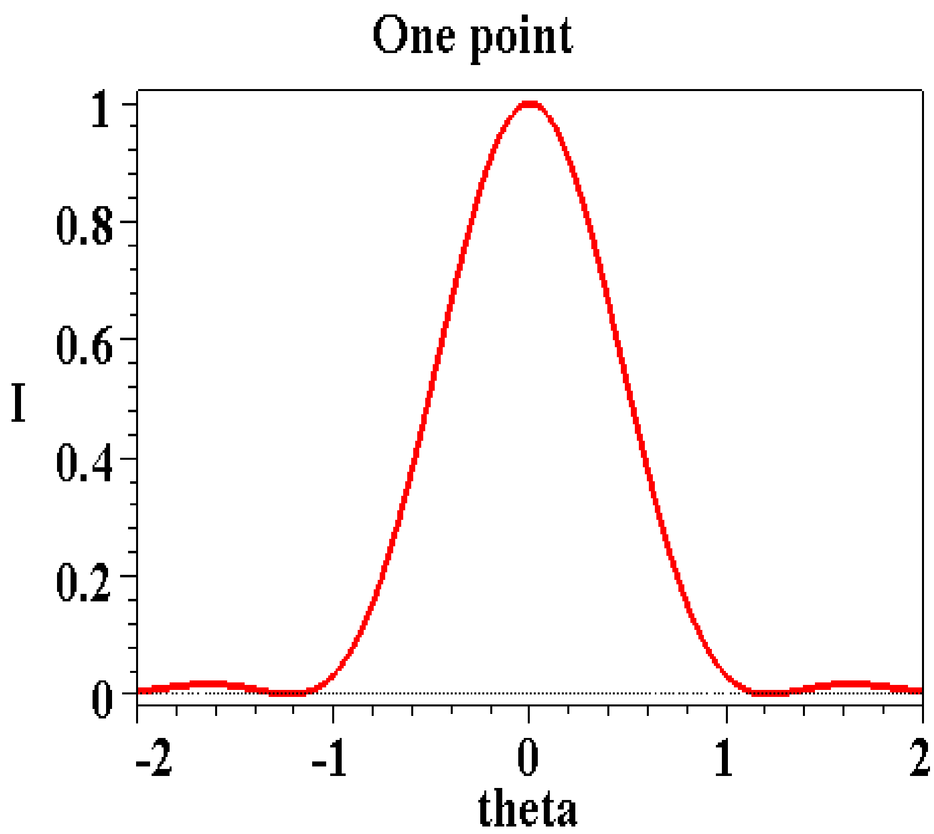
The image is a "degraded" version of the object.
Nevertheless, given a fixed entrance pupil diameter, the Airy PSF is the minimum possible degradation.
We say that the system is "***diffraction limited***"

$$I(\vec{\alpha}) = \int O(\vec{\beta})\, P_0(\vec{\alpha}-\vec{\beta})\, d\vec{\beta} = O \otimes P_0$$

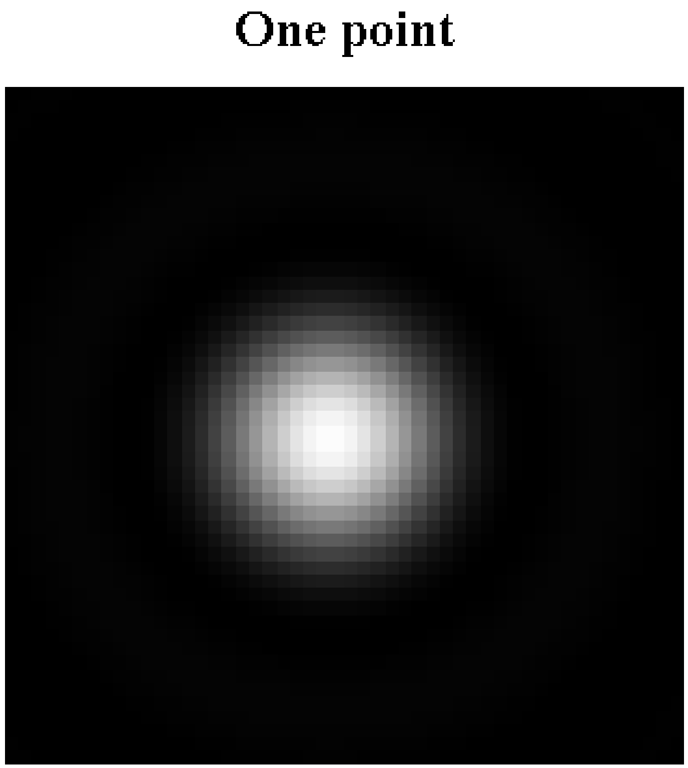
$$\mathrm{FT}\mathit{(obj)}\,\mathrm{FT}\mathit{(psf)} = \mathrm{FT}\mathit{(imm)}$$


Galactic Center / 2.2 microns
13"x13" Field. 15 minutes exposure.
Without Adaptive Optics compensation
0.57" Seeing
With Adaptive Optics compensation
0.13" Full Width at Half Maximum
Copyright CFHT. 1996.

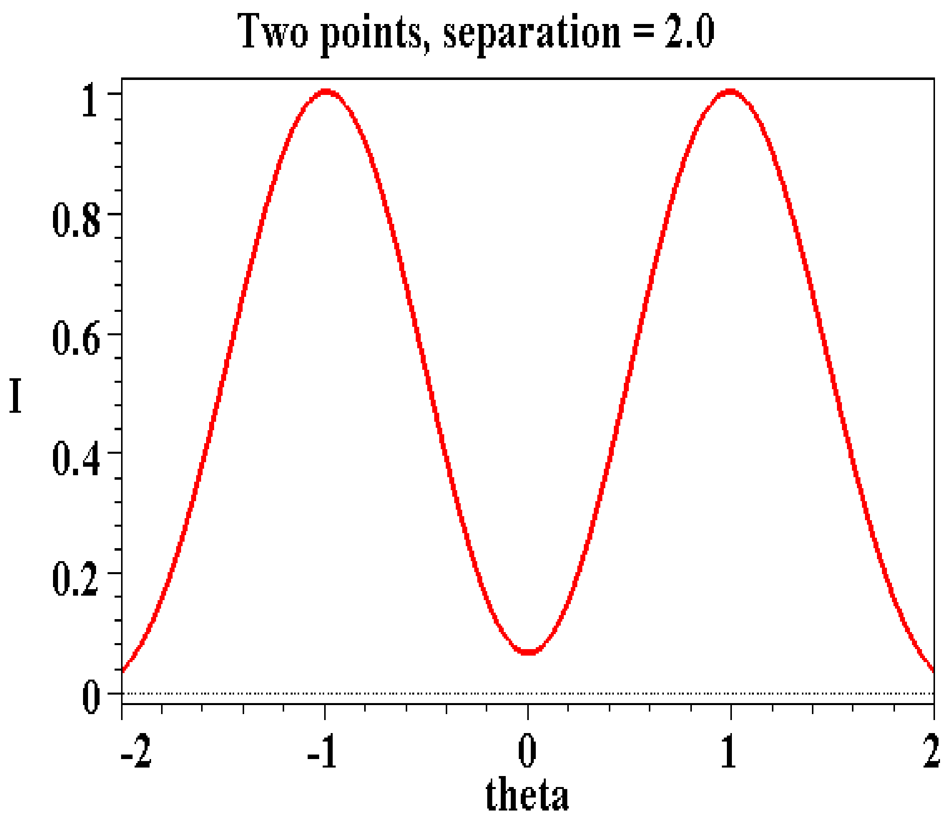
Single Star

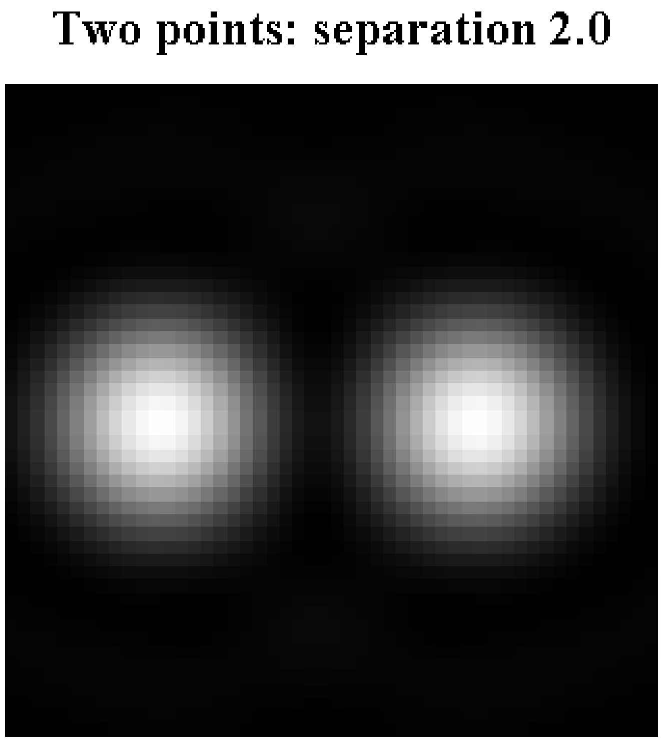


Units in multiples of λ/D



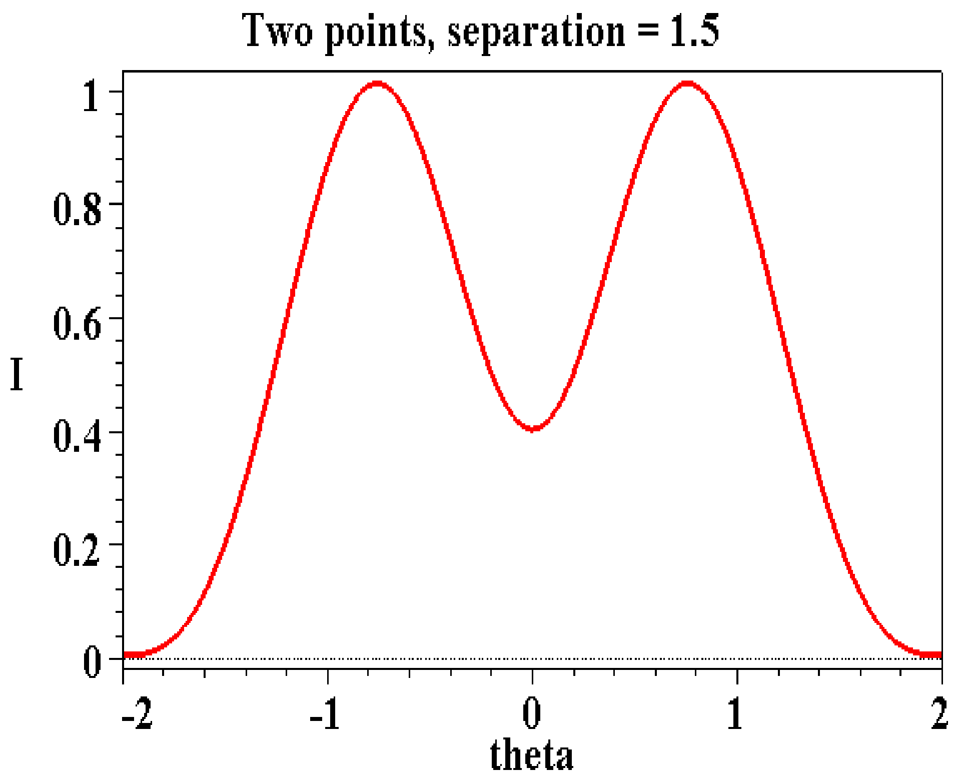
Two Stars: Separation = 2.0

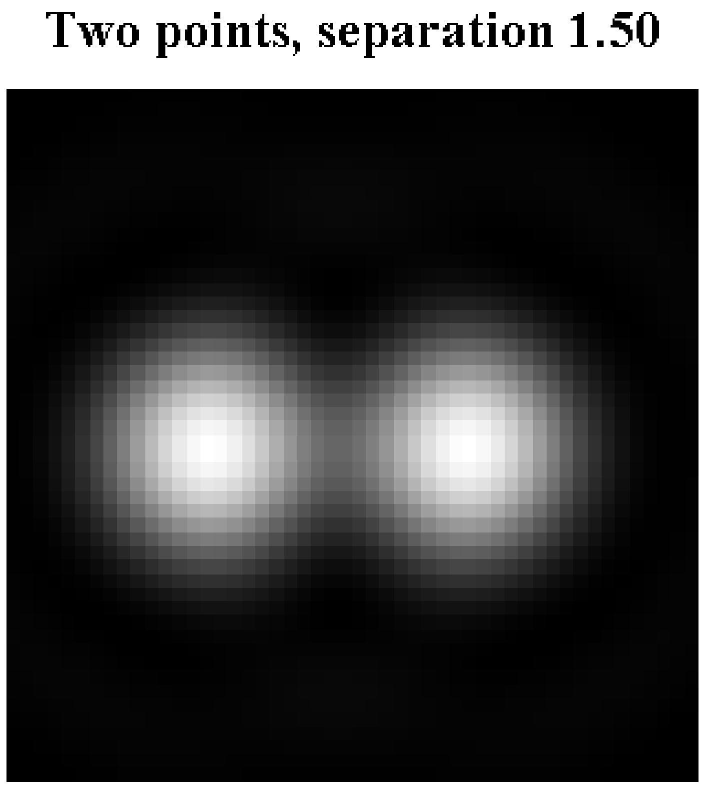




Units in multiples of λ/D

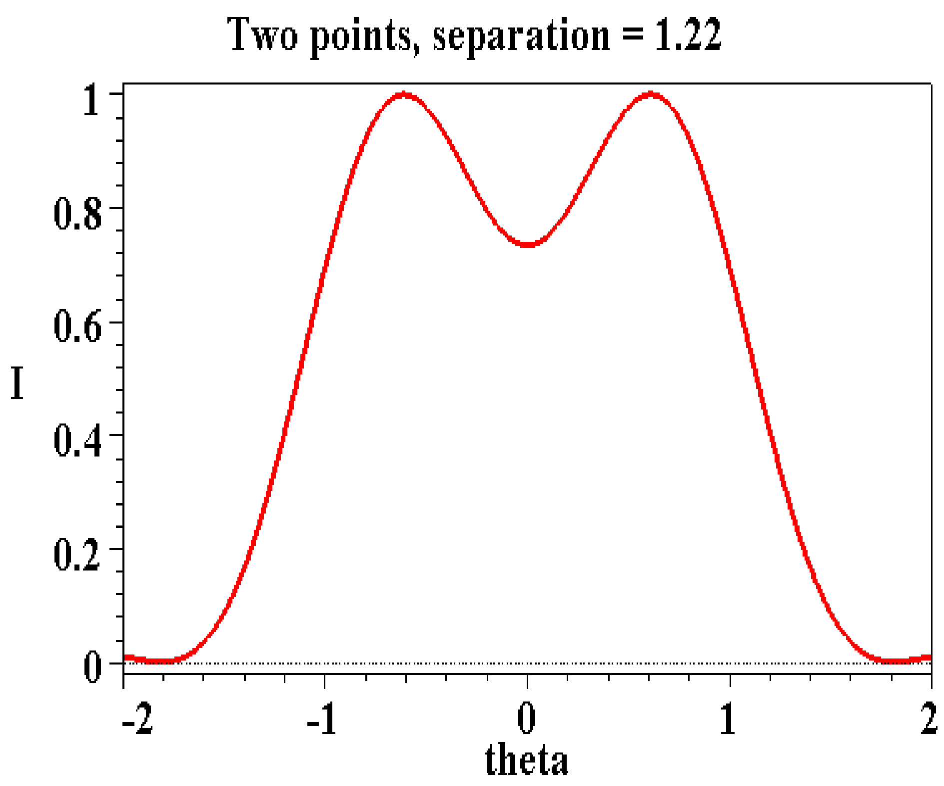
Two Stars: Separation = 1.5

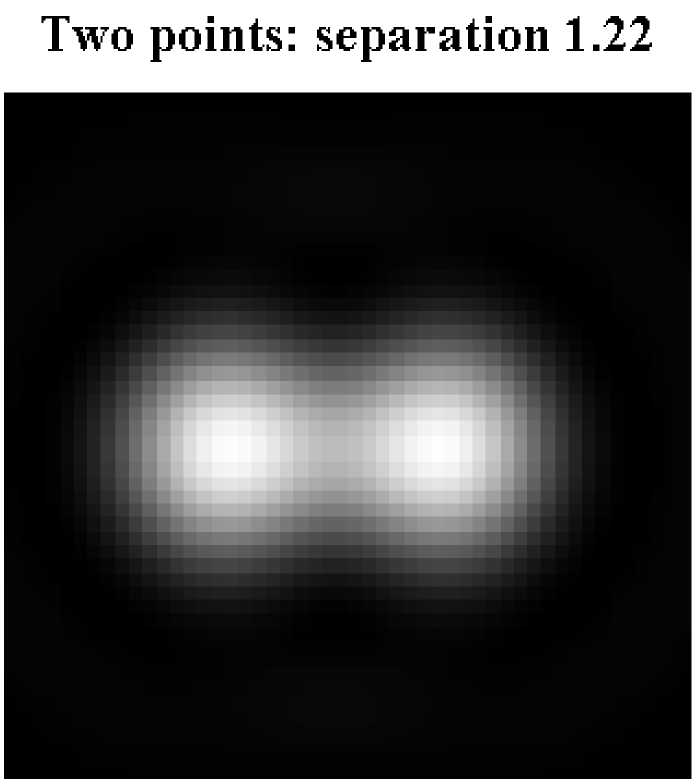




Units in multiples of λ/D

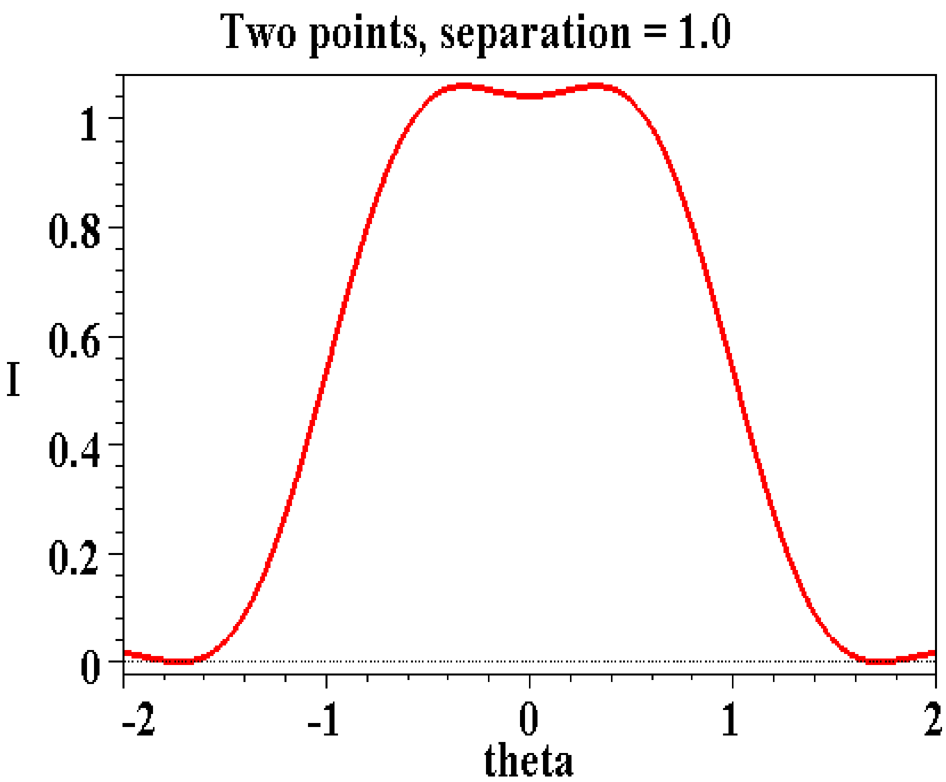
Two Stars: Separation = 1.22

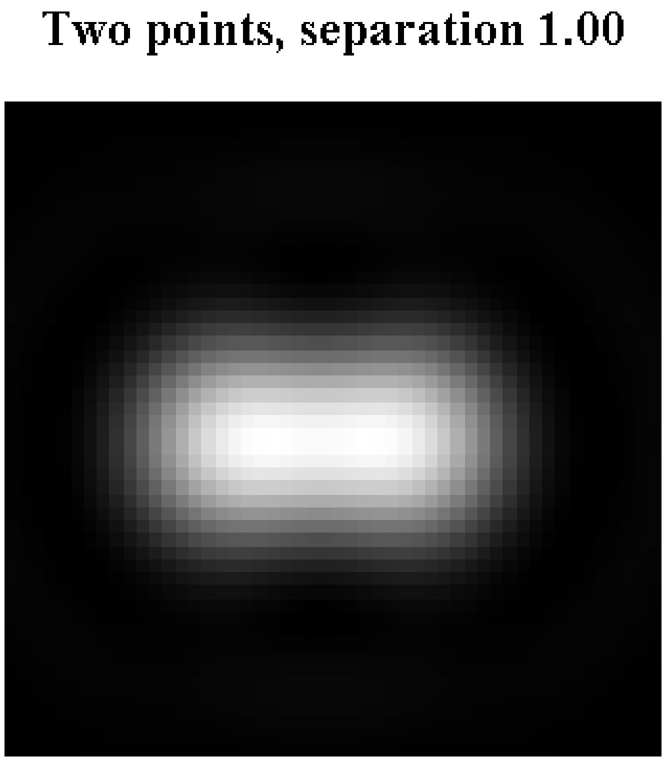




Units in multiples of λ/D

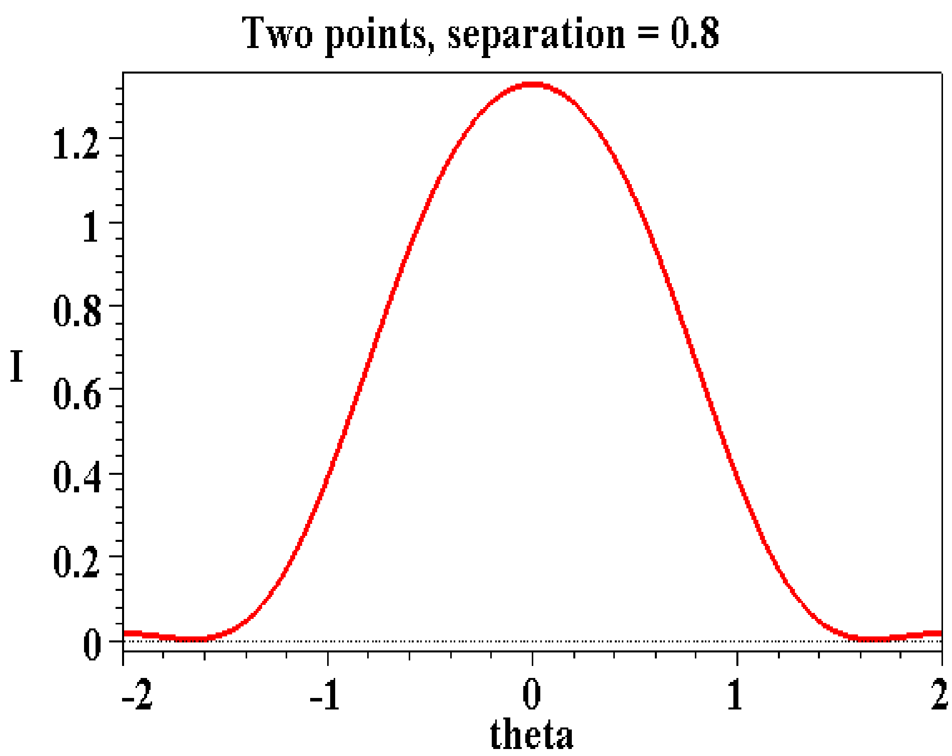
Two Stars: Separation = 1.0

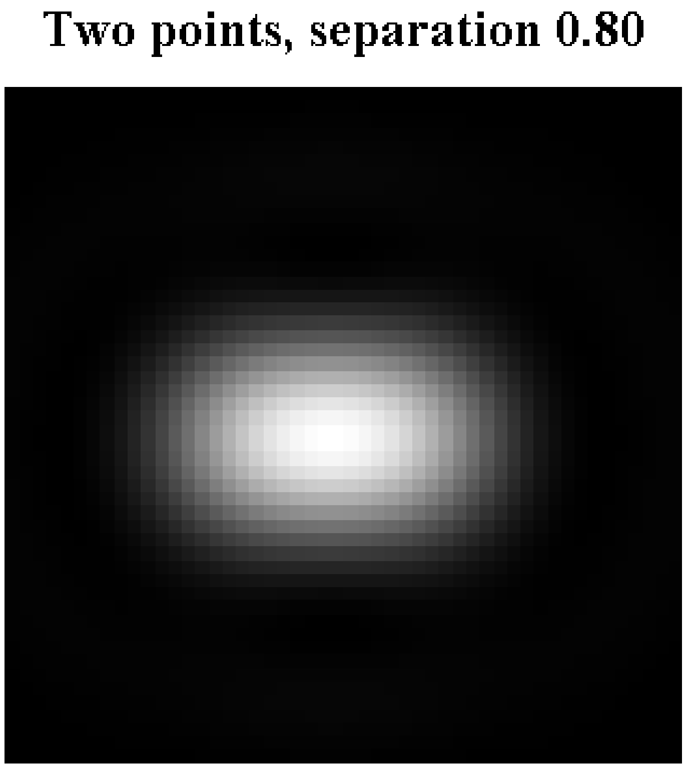




Units in multiples of λ/D

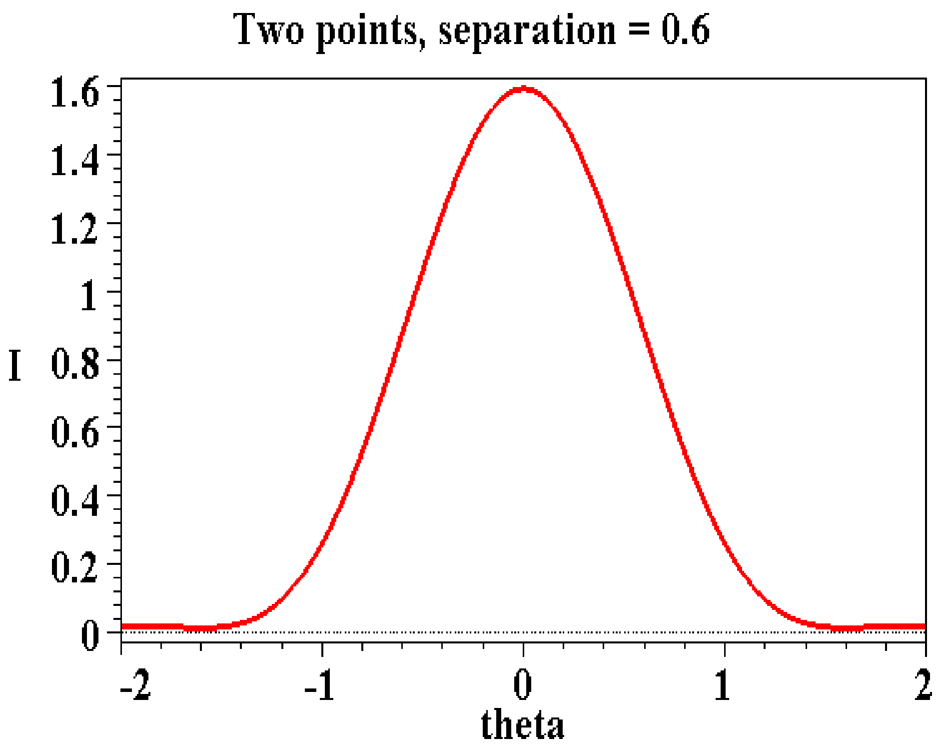
Two Stars: Separation = 0.8



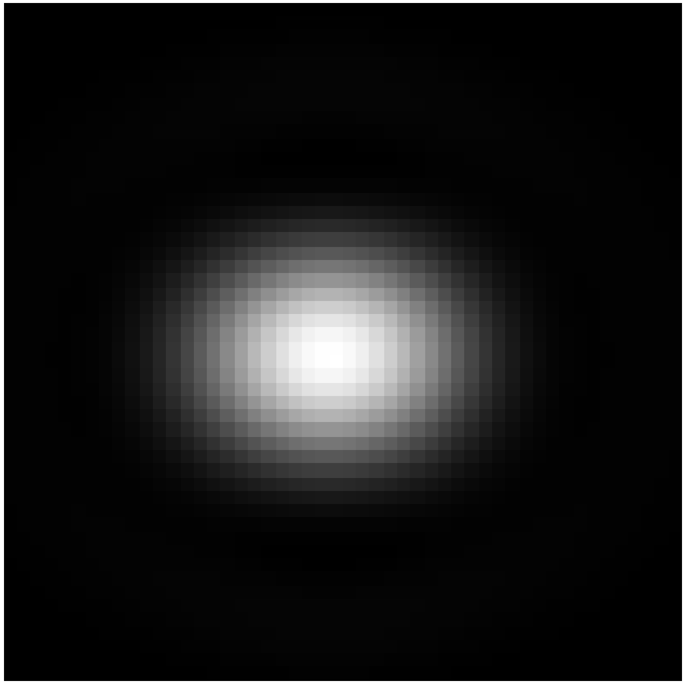


Units in multiples of λ/D

Two Stars: Separation = 0.6

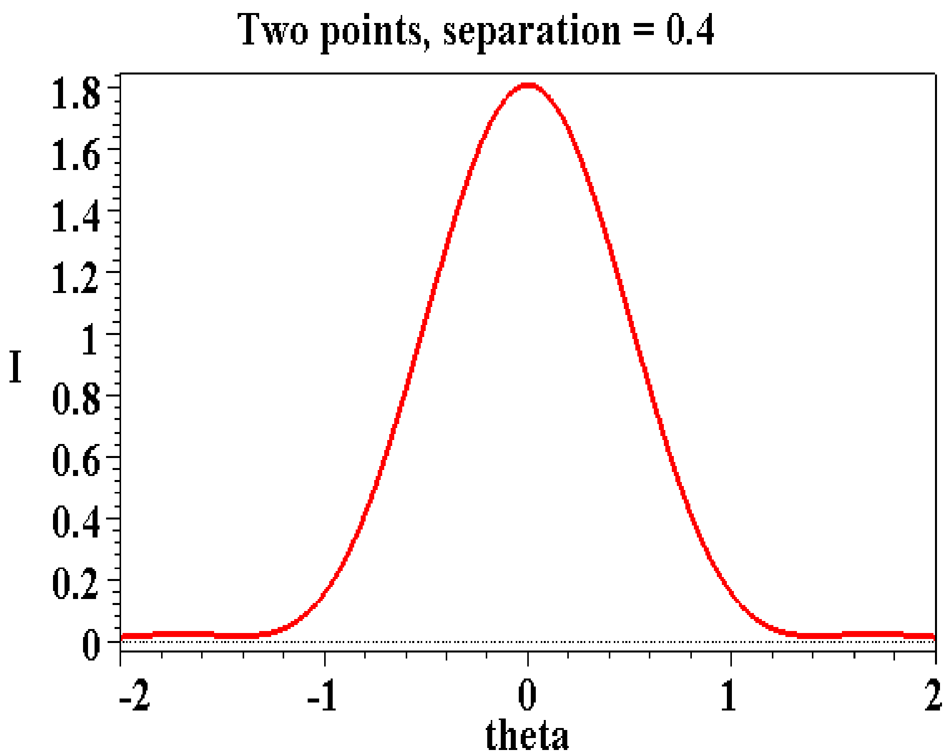


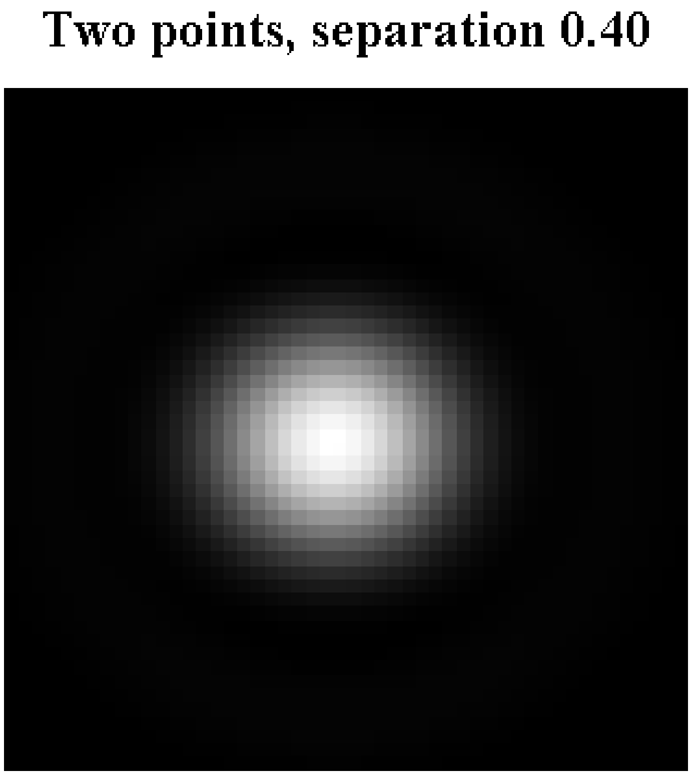
Two points, separation 0.60



Units in multiples of λ/D

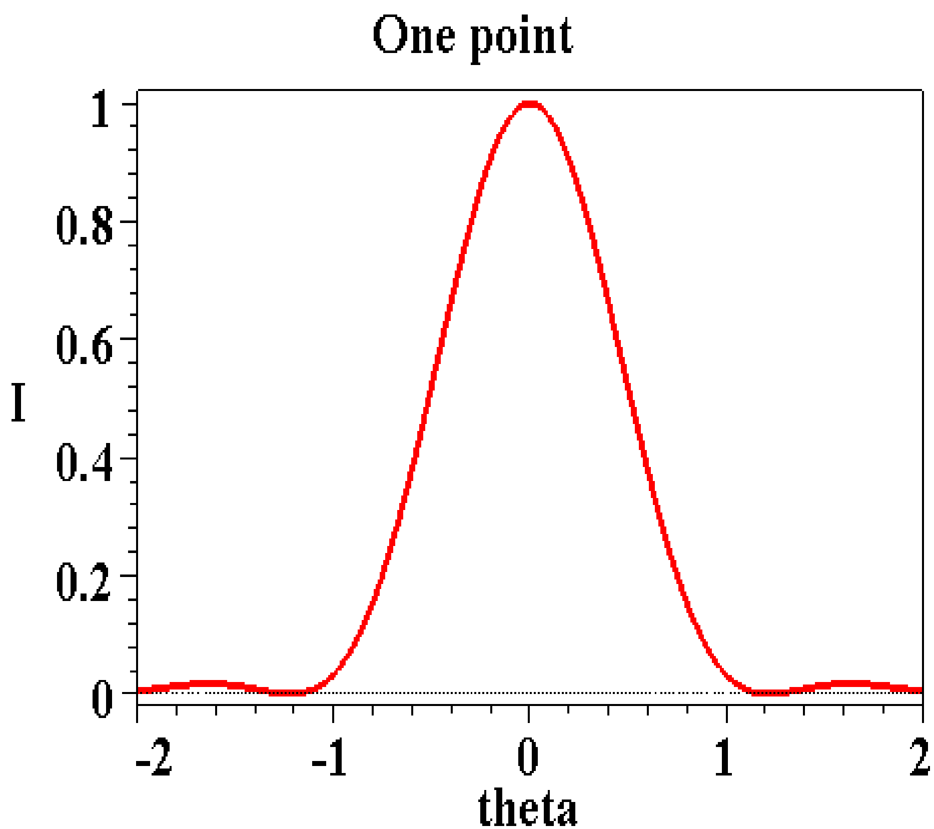
Two Stars: Separation = 0.4



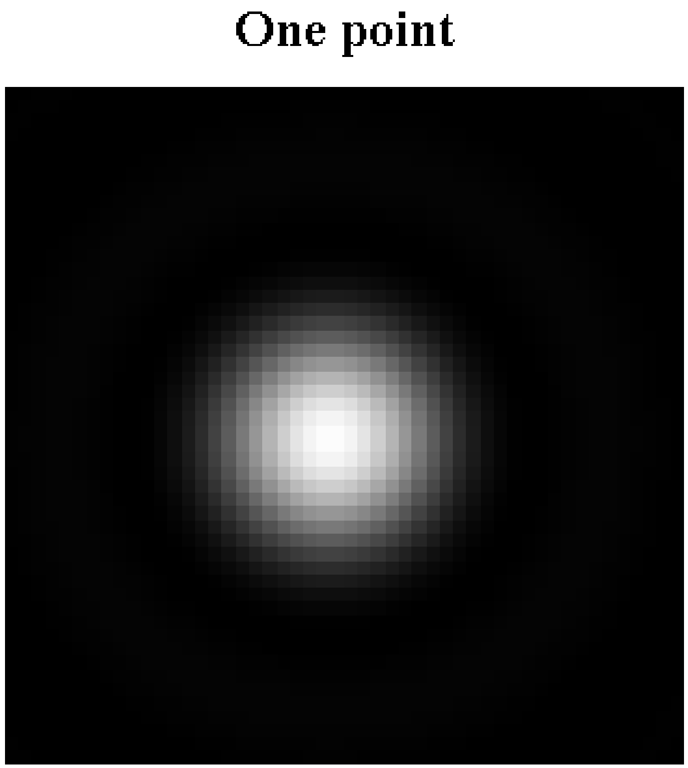


Units in multiples of λ/D

Single Star



Units in multiples of λ/D



Optical Transfer Function

The performance of an optical system is quantified by a *point spread function* or a *transfer function*.

In astronomy we image spatially incoherent objects, so the *intensity* point spread function is used:

$$
\begin{aligned}
I(\xi,\eta) &= \mathcal{O}(\xi,\eta) \otimes \mathcal{P}(\xi,\eta) \\
&= \iint \mathcal{O}(\xi',\eta')\mathcal{P}(\xi'-\xi,\eta'-\eta)\,\mathrm{d}\xi'\,\mathrm{d}\eta'
\end{aligned}
$$

In terms of spatial frequencies (Fourier space)

$$\tilde{I}(u,v) = \tilde{\mathcal{O}}(u,v)\mathcal{H}(u,v)$$

$\mathcal{H}$ is the optical transfer function (OTF), which is given by the *autocorrelation* of the *pupil function*.

$$\mathcal{H}(u,v) = P(\lambda fu,\lambda fv) \star P(\lambda fu,\lambda fv)$$

The Modulation transfer function (MTF)

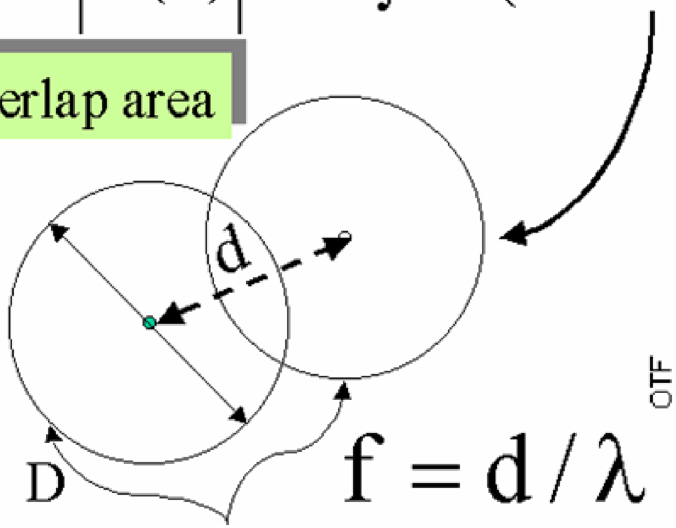
$$\tilde{I}(\vec{f}) = \tilde{O}(\vec{f}) \bullet \tilde{P}(\vec{f})$$

Optical Transfer function

$$\text{MTF} = \left|\tilde{P}(\vec{f})\right| \propto \int W(\vec{r}+\vec{d})W(\vec{r})d\vec{r}$$

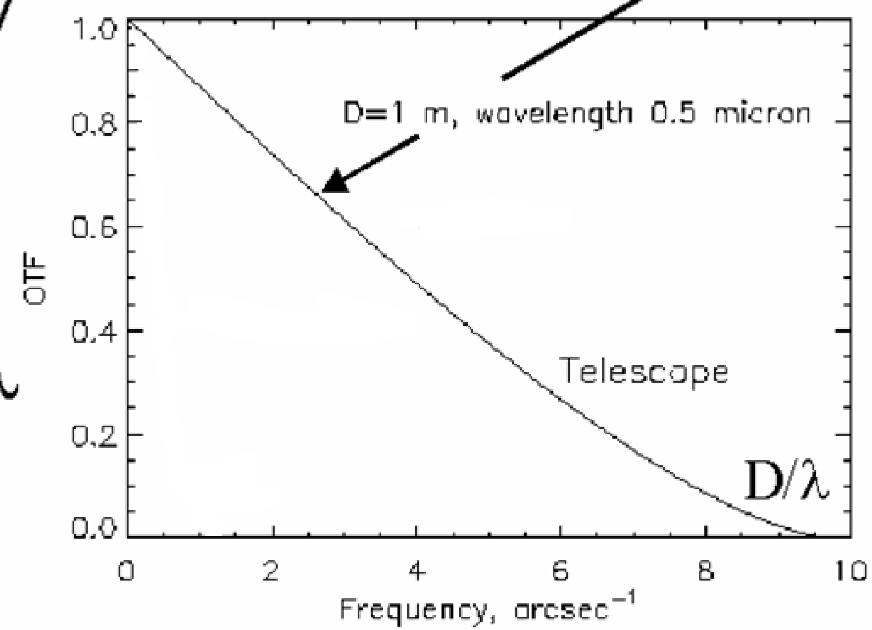
Limit MTF

Overlap area



$$f = d/\lambda$$

MTFs

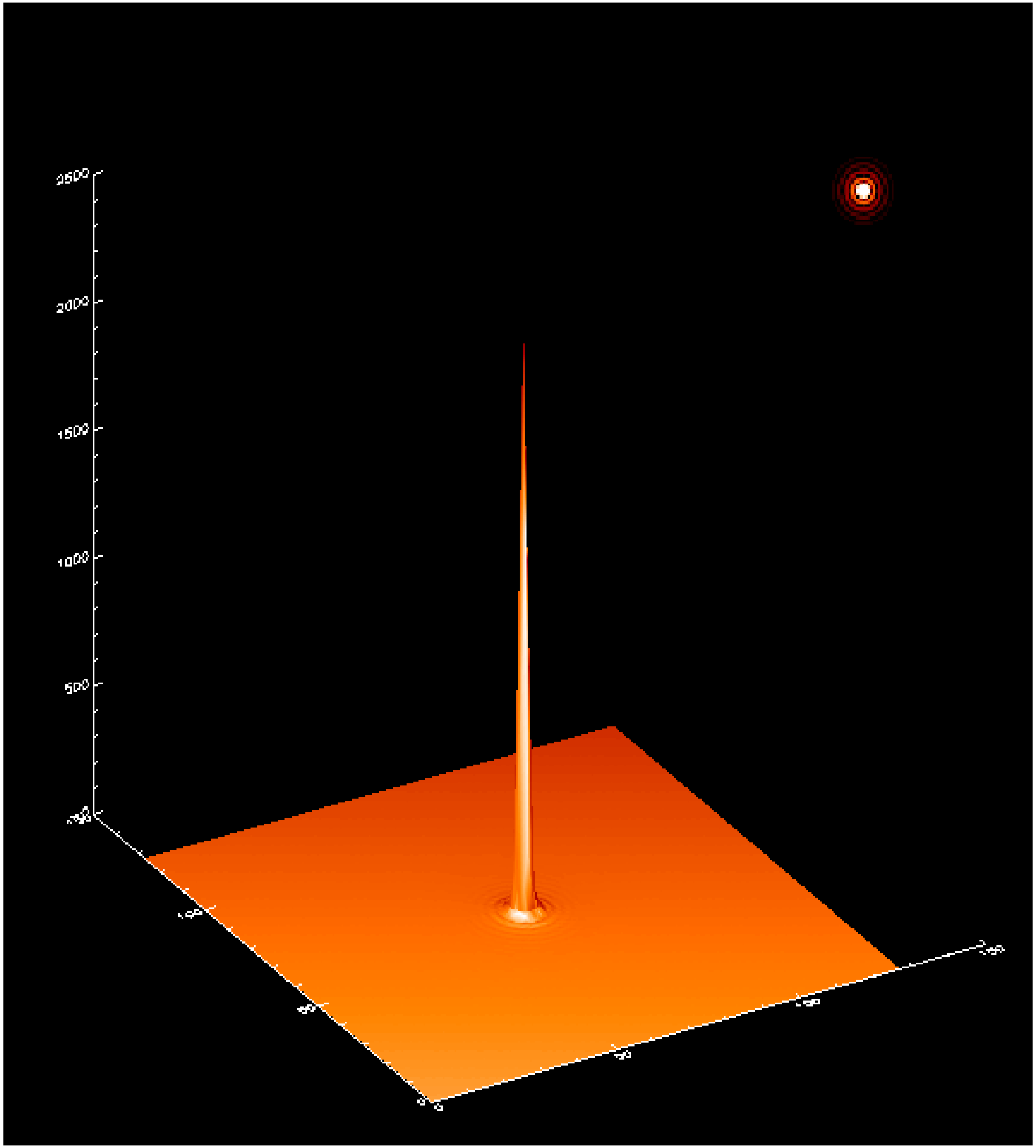


W=Pupil function

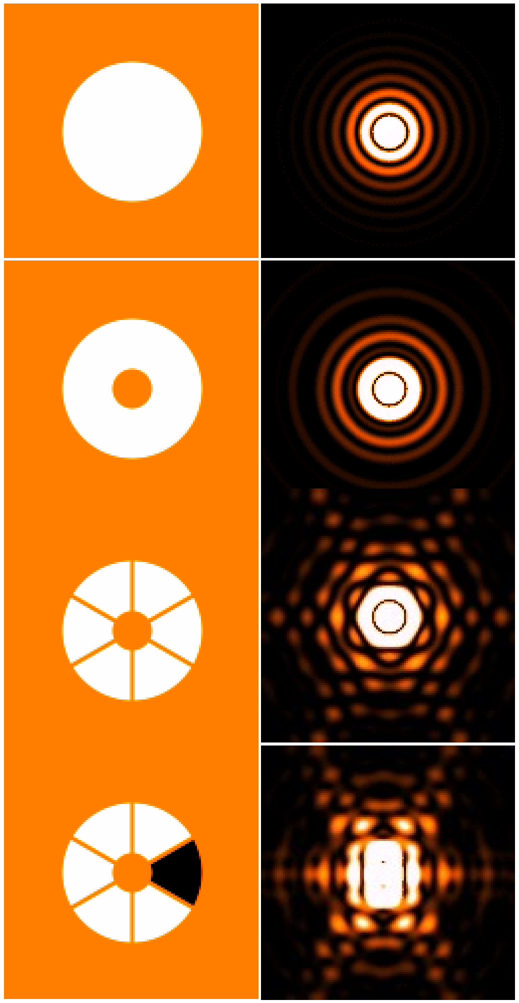
- $r < D/2$

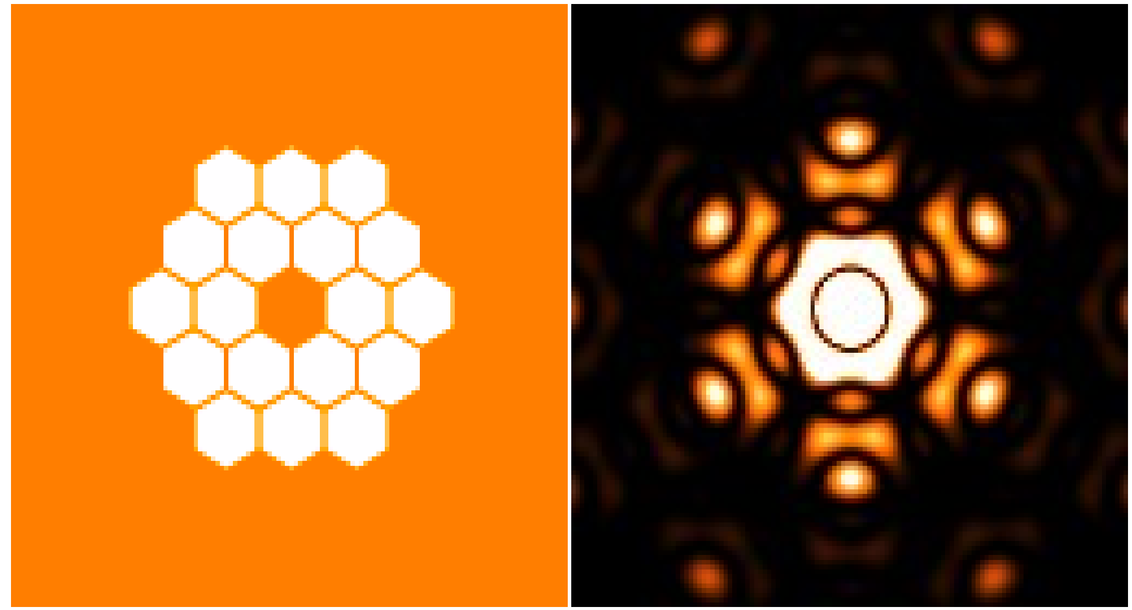
0 $\quad r > D/2$

- Pupil function → OTF
  - » autocorrelation
- OTF → MTF
  - » |OTF|=MTF;
  - » Im(OTF)=PTF ≡ 0. ***(without any aberration)**
- OTF → PSF
  - » Inverse Fourier Transform
- PSF + object → image
  - » Convolution



PSF of a 'perfect' 47 cm telescope





http://www.kusastro.kyoto-u.ac.jp/~iwamuro/Kyoto3m/psf.html

About the Notation:

$\boldsymbol{\rho}$ e $\boldsymbol{u}$ are conjugated variables

PSF = $s(x,y;x',y';t) = s(\boldsymbol{\rho};\boldsymbol{\rho}';t)$

PSF = $s(\boldsymbol{\rho} - \boldsymbol{\rho}')$

ISOPLANATISM+*TEMPORAL INVARIANCE*

$\downarrow$

OTF = FT [PSF]= $S(u,v)$

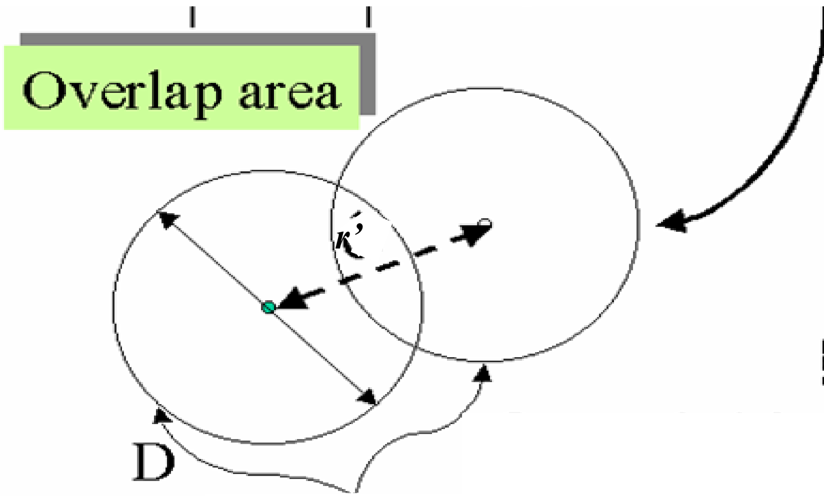
OTF = FT[$s(\boldsymbol{\rho}-\boldsymbol{\rho}')$]= $S(\boldsymbol{u})$

$\Delta u=1/(N\Delta x)$

$\Delta v=1/(N\Delta y)$

$$S(\vec{u}) \propto \int W(\vec{r})\, W^*(\vec{r} - \lambda f \vec{u}) \cdot d\vec{r}$$

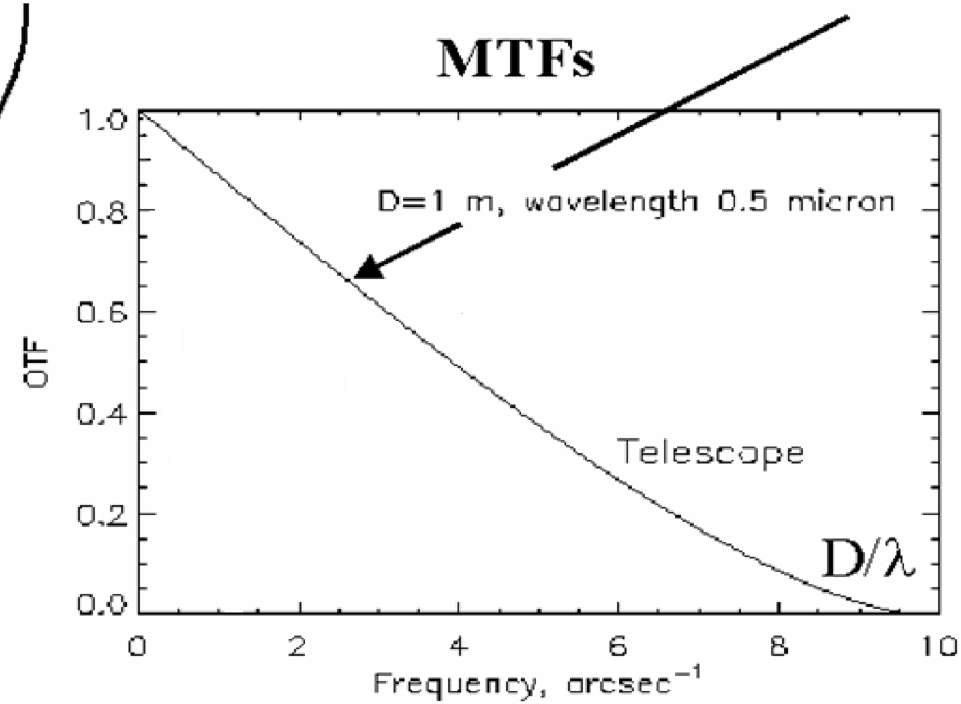
$$S(|\vec{u}|) = \frac{2}{\pi} \cdot \left[\arccos\left(\frac{\lambda f}{D}|\vec{u}|\right) - \frac{\lambda f}{D}|\vec{u}| \sqrt{1 - \left(\frac{\lambda f}{D}|\vec{u}|\right)^2} \right]$$



W = Pupil Function

• r < D/2

0 r > D/2



OTF = FT [PSF] = S(u, v)

OTF = FT[s($\boldsymbol{r}$-$\boldsymbol{r}'$)] = S($\boldsymbol{u}$)

$$S\left(|\vec{u}|\right)=\frac{2}{\pi}\cdot\left[\arccos\left(\frac{\lambda f}{D}|\vec{u}|\right)-\frac{\lambda f}{D}|\vec{u}|\sqrt{1-\left(\frac{\lambda f}{D}|\vec{u}|\right)^2}\right]$$

$$\Downarrow$$

$$s\left(|\vec{\rho}|\right)=\frac{1}{\pi|\vec{\rho}|}\cdot J_1\left(\frac{D\pi}{\lambda f}|\vec{\rho}|\right)$$

Airy function

What is the relation between *r'* and *u* on the pupil plane?

D = telescope diameter
f = telescope focal length
λ = wavelength

$$\vec{u} = \frac{\vec{r'}}{\lambda f}$$

$$|\vec{r'}| < \frac{D}{2} \rightarrow |\vec{u}| < \frac{D}{2\lambda f}$$

$$\frac{u_{cutoff}}{2}$$

Working in the ***u-space*** means to rescale the telescope pupil.

Radius of the Pupil in the *u-space*

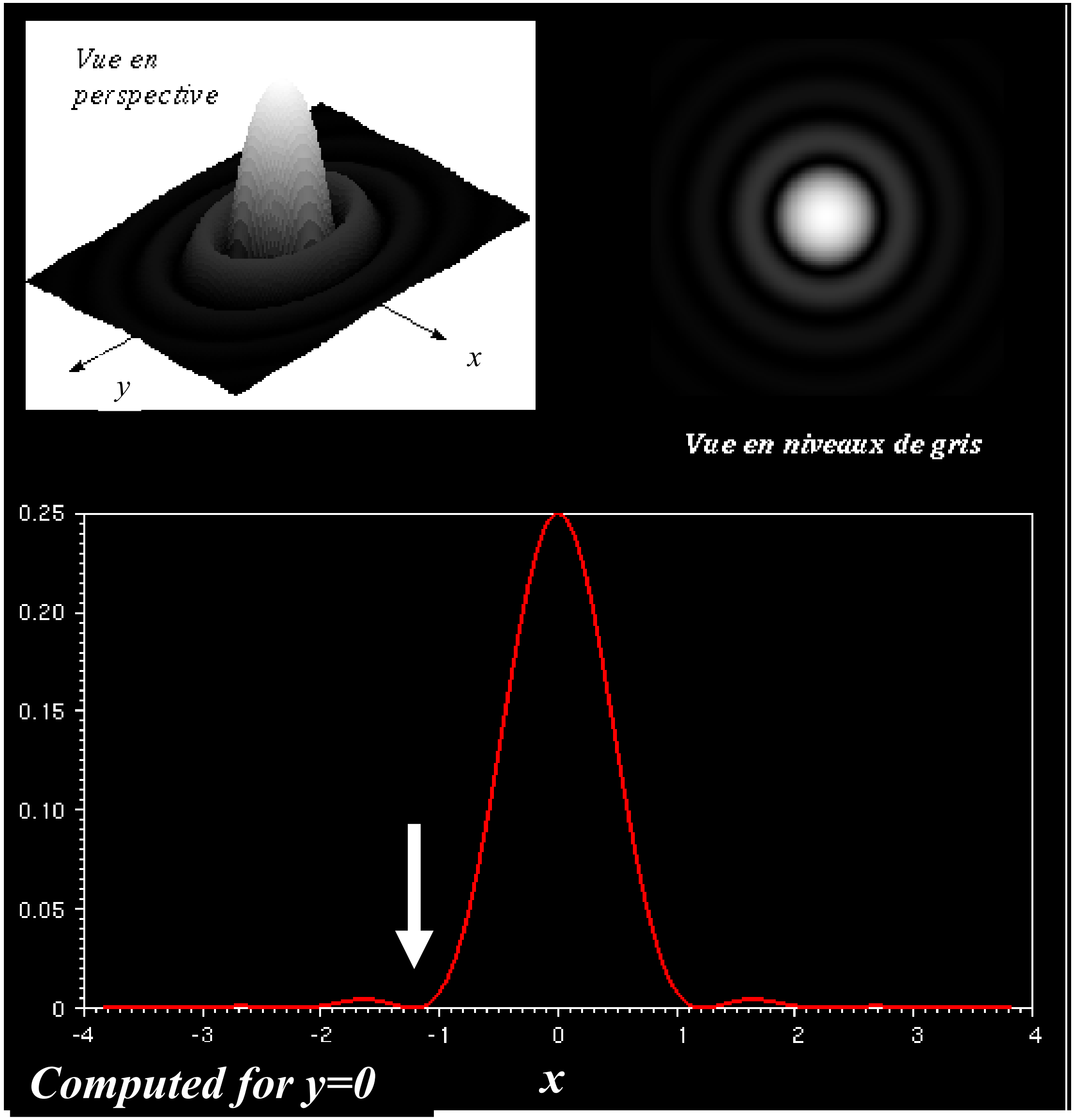
$$r'_p = \frac{D}{2\lambda f}[cm^{-1}] = \frac{\dfrac{D}{2\lambda f}}{\dfrac{1}{N\Delta x}}[pixels] = \frac{\dfrac{D}{\lambda}}{\dfrac{2}{N\dfrac{\Delta x}{f}}}[pixels]$$

$$r'_p = \frac{v_{cutoff}[rad^{-1}]}{2\Delta u[rad^{-1}]}[pixels]$$

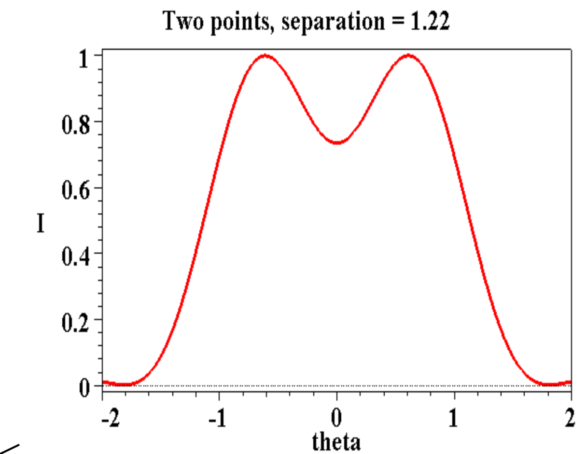
ρ on the focal plane

r on the pupil plane

u on the conjugated pupil plane

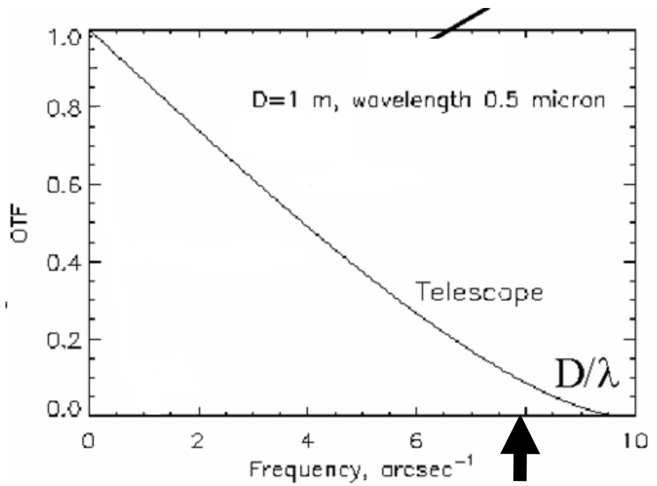


$$s\left(|\vec{\rho}|\right)=\frac{1}{\pi}\cdot\left[\frac{J_1\left(\frac{D\pi}{\lambda f}|\vec{\rho}|\right)}{|\vec{\rho}|}\right]$$

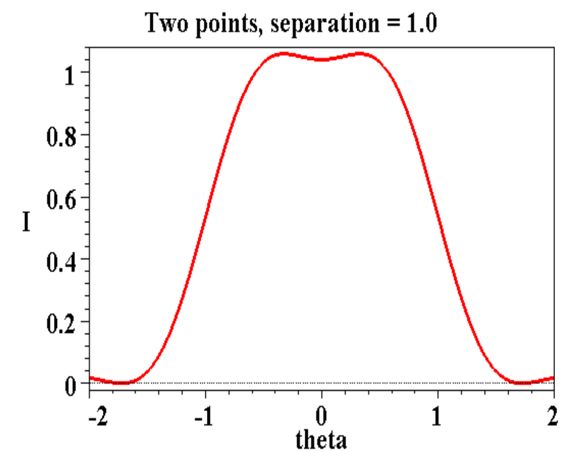


$$\rho_1=3.83\cdot\frac{\lambda f}{D\pi} \longrightarrow \theta_1=1.22\cdot\frac{\lambda}{D}$$

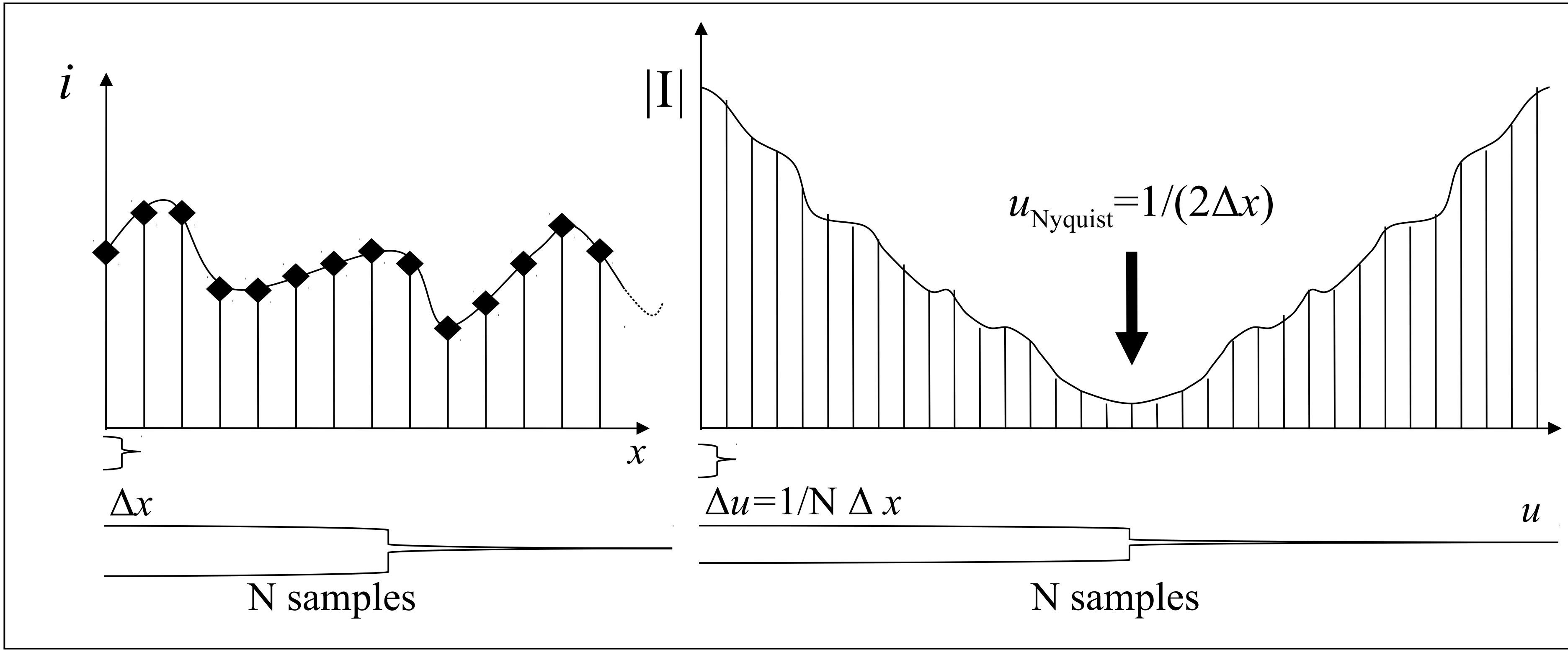
Rayleigh's Criterion



$$\theta_{\lim}=\frac{\lambda}{D}$$



Nyquist frequency



At least two samples per period for the highest frequency present in the signal.

- Otherwise... the improperly sampled (high) frequencies will appear as properly sampled (lower) frequencies

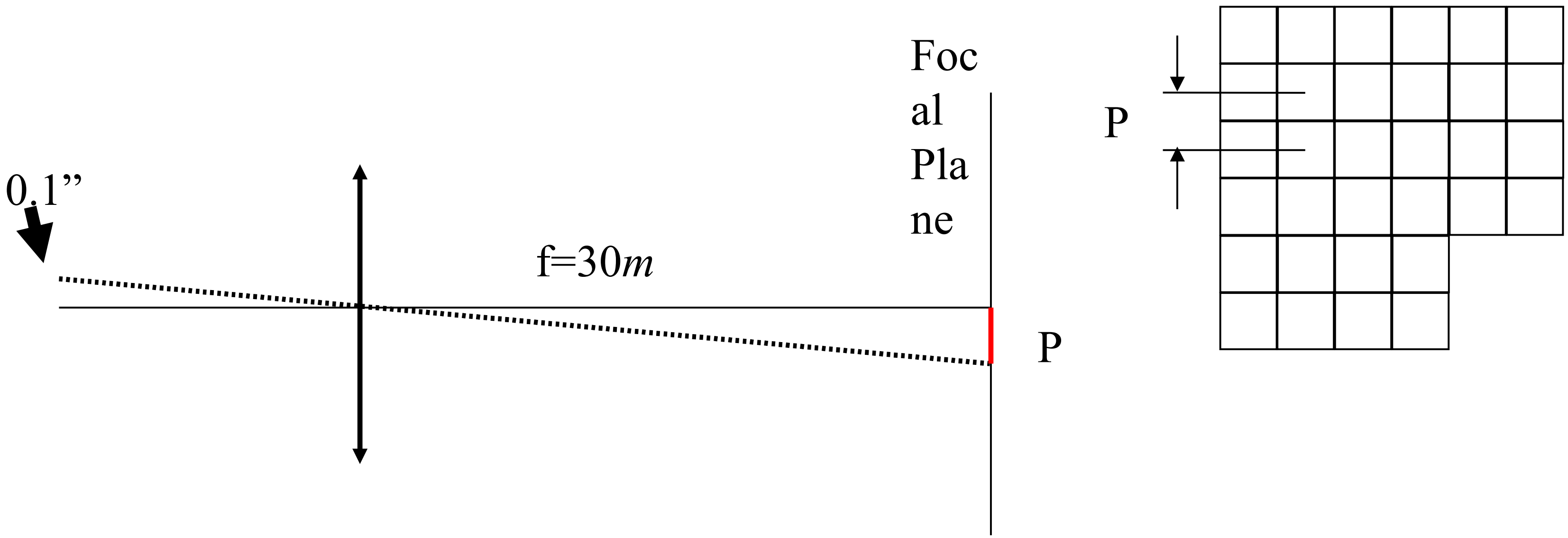
A couple of problems:

I. From a telescope specifications, we know that the highest angular resolution it can obtain is **0.2”**. The telescope focal length is **30 *m***. Compute the maximum dimension of the detector pixels necessary to utilize the full telescope resolution.

II. Compute the PSF at λ=450.7nm of an optical system (diffraction limited) that has D=90cm, f=57m on a 300×300 CCD with 0.075” pixel scale.

Problem I

From a telescope specifications, we know that the highest angular resolution it can obtain is **0.2”**. The telescope focal length is **30 *m***. Compute the maximum dimension of the detector pixels necessary to exploit all the telescope resolution.



Problem II

Compute the PSF at λ=450.7*nm* of an optical system (diffraction limited) that has *D*=90*cm*, *f*=57*m* on a 300×300 CCD with 0.075” pixel scale.

Solution N.1: USE the Airy function

$$s(|\vec{\rho}|)=\frac{1}{\pi|\vec{\rho}|}\cdot J_1\left(\frac{D\pi}{\lambda f}|\vec{\rho}|\right)$$

Solution N.2: Numerical approach

$$r_p'=\frac{D}{2\lambda f}[cm^{-1}]=\frac{\frac{D}{2\lambda f}}{\frac{1}{N\Delta x}}[pixels]=\frac{\frac{D}{2\lambda}}{\frac{1}{N\frac{\Delta x}{f}}}[pixels] \quad + \quad \ldots$$

Solution 2

- $\text{Pixel}_{rad} = (\text{Pixel\_scale}) \times\pi /180/3600$ [rad]
- $u_{cutoff} = D/\lambda$ [rad^{-1}]
- $\Delta u = 1/(\text{N}_{pixels} \times \text{Pixel}_{rad})$ [rad^{-1}]
- $r_p = u_{cutoff} /(2\Delta u)$ [pixels]

- Define the pupil function $W(r)$: a disk with radius r_p onto a 300×300 matrix
- Get the OTF computing the autocorrelation of $W(r)$
- Get the PSF with an inverse Fourier transform of the OTF